§3 $m(E) \leq +\infty$

In §2 we have treated $\int_E f$ $\forall f \in \mathcal{B}_0(E)$, the class of all bounded measurable functions on E vanishing outside some subset of E of finite measure. This section we will deal with the integral for measurable functions bounded below by zero:

$\mathcal{M}^+(E)$ (or $\mathcal{MF}^+(E)$) $= \{f$: measurable, $f=0$ on $\mathbb{R}\setminus E$, & $f \geq 0\}$

Define, $\forall f \in \mathcal{MF}^+(E)$ (or $f = f'$ a.e on E for some $f' \in \mathcal{MF}^+(E)$)

$$\left.\begin{aligned} \int_E f &\overset{\text{def}}{=} \sup\left\{\int_E g : g \in \mathcal{B}_0(E), g \leq f \text{ on } E\right\} \\ &= \sup\left\{\int_E h : h \in \mathcal{B}_0(E),\ 0 \leq h \leq f \text{ on } E\right\} \quad (\leq +\infty). \end{aligned}\right\} (*)$$

To see the last equality, one notes that, if $g \in \mathcal{B}_0(E)$ and $g \leq f$ on E then $h := g \vee 0 \in \mathcal{B}_0(E)$ and $0 \leq g \vee 0 \leq f$ on E. Then $\int_E g \leq \int_E (g \vee 0) \leq$ the last sup in the displayed lines $(*)$

The other direction ($\sup\{\int_E g : \cdots\} \geq \sup\{\int_E h : \cdots\}$) is evident.

Theorem 1. $\mathcal{MF}^+(E) \ni f \mapsto \int_E f$ is monotone (w.r.t. a.e order), and is "linear" as far as it makes sense. More precisely,

(i) $f, g \in \mathcal{MF}^+(E)$ with $f = g$ a.e on $E \Rightarrow \int_E f = \int_E g$

(ii) $f, g \in \mathcal{MF}^+(E)$ with $f \leq g$ a.e on $E \Rightarrow \int_E f \leq \int_E g$

(iii) Let $f \in \mathcal{MF}^+(E)$ and $A \subseteq E$ measurable. Then

$\int_A f \overset{\text{def}}{=} \sup\{\int_A h : 0 \leq h \leq f,\ h \in \mathcal{B}_0(A) (\subset \mathcal{B}_0(E))\} = \int (f \chi_A)$

$\int_A f = \sup\{\int_A h : h \le f, \ h \in \ldots (= \ldots)\} = \int_E f\chi_A$

(iv) $f \in \mathcal{MF}^+(E)$ & $0 \le \alpha \in \mathbb{R} \Rightarrow \alpha f \in \mathcal{MF}^+(E)$ & $\int_E (\alpha f) = \alpha \int_E f$

(v) $f_1, f_2 \in \mathcal{MF}^+(E) \Rightarrow f_1 + f_2 \in \mathcal{MF}^+(E)$ & $\int_E (f_1+f_2) = \int_E f_1 + \int_E f_2$

(vi) $f \in \mathcal{MF}^+(E)$, $\int_E f < +\infty \Rightarrow f(x) < +\infty$ a.e $x \in E$.

(vii) $f, g \in \mathcal{MF}^+(E)$, $f \le g$ on E & $\int_E f < +\infty$

$\Rightarrow g - f \in \mathcal{MF}^+(E)$ (well-defined a.e on E by (vi)), $\int_E (g-f) = \int_E g - \int_E f$.

proof. We only prove (v), (vi) & (vii) (other parts are Ex.).

(v). By definitions, easy to show LHS $\ge$ RHS. Conversely let $0 \le h \le f_1 + f_2$ on E, and $h \in \mathcal{B}_0(E)$ (so $\exists H \subseteq E$ of finite mea and $M \in \mathbb{R}$ s.t. $0 \le h \le M$ on E and $h = 0$ outside H. Let $h_1 = h \wedge f_1$ and $h_2 = h - h_1$. Then $0 \le h_1 \le h, f_1$ on E, $h_1 \le M$ on E and $h_1 = 0$ outside H (so $h_1 \in \mathcal{B}_0(E)$); moreover $h_2 \in \mathcal{B}_0(E)$, $h = h_1 + h_2$ and $h_2 = h - (h \wedge f_1) = (h-h) \vee (h - f_1) \le f_2$. Consequently $\int_E h = \int_E h_1 + \int_E h_2 \le \int_E f_1 + \int_E f_2$. Taking the sup over all such h, one has $\int_E (f_1+f_2) \le \int_E f_1 + \int_E f_2$.

This proves (v). Note then that we have another kind of linearity (see (iii)): if $A_1 \cup_0 A_2 \subseteq E$ with

$A_1, A_2 \in \mathfrak{M}$ then $\int_{A_1 \cup_0 A_2} f = \int_{A_1} f + \int_{A_2} f \quad \forall f \in \mathcal{MF}^+(E)$

and monotonicity if $A \subseteq B \subseteq E$ with $A, B \in \mathfrak{M}$ then $\int_A f \leq \int_B f$ (as $\int_B f = \int_A f + \int_{B \setminus A} f$), regardless $\int_B f = +\infty$ or finite. Hence (vi) holds:

$$n \cdot m(\Delta_\infty) = \int_{\Delta_\infty} n \leq \int_{\Delta_\infty} f \leq \int_E f < +\infty \quad \forall n \in \mathbb{N}$$

$$\Longrightarrow m(\Delta_\infty) = 0,$$

where $\Delta_\infty := \{x \in E : f(x) = +\infty\}$.

(vii): By (v) and (vi).

(Fatou's Lemma)

Theorem 2. Let $\{f_n\}$ be a seq in $\mathcal{MF}^+(E)$ convergent a.e. on E to $f \in \mathcal{MF}^+(E)$. Then

$$\int_E f \leq \liminf_{n \to \infty} \int_E f_n \qquad (\#)$$

Proof. Wlog. $f_n(x) \to f(x) \;\; \forall x \in E$. Let $h \in \mathcal{B}_0(E)$ (say h is bounded by $M \in \mathbb{R}$ & $h = 0$ vanishes outside H of finite measure) be s.t. $0 \leq h \leq f$. Need to show that

$$\int_E h \leq \liminf_{n \to \infty} \int_E f_n .$$

To do this, let $h_n := h \wedge f_n$. Then

$h_n \le f_n$ and $h_n \in \mathcal{B}_0(E)$ so $\int_E h_n \le \int_E f_n$

and all h_n bounded by M and vanishes outside the same H of finite measure.

By the (extended) BC Theo $\lim_n \int_E h_n = \int_E (\lim h_n) = \int_E (h \wedge f) = \int_E h$ and so

$$\int_E h = \lim_n \int_E h_n = \liminf_n \int_E h_n \le \liminf_n \int_E f_n,$$

as wished to show. Here $\lim_n h_n$ is meant to be the function s.t.

$$(\lim_n h_n)(x) = \lim_n h_n(x) = \lim_{n\to\infty} (h(x) \wedge f_n(x))$$
$$= \lim_{n\to\infty} (h(x) \wedge \lim_n f_n(x)) = \lim_{n\to\infty} (h(x) \wedge f(x)) = h(x).$$

Th3. Monotone Conv. Th. Let $\{f_n\}$ be a increasing seq in $MF^+(E)$ convergent a.e. to $f \in MF^+(E)$. Then $\lim_n \int_E f_n = \int_E f$.

proof. Note that $f_n \le f$ a.e. on E and so $\int_E f_n \le \int_E f$ and it follows from Fatou's Lemma that

$$\int_E f \le \liminf_n \int_E f_n \le \limsup_n \int_E f_n \le \int_E f$$

so equal for all.

Th4 (infinite sums for functions & for domains).

(i) $\int (\sum^{\infty} u_n) = \sum^{\infty} \int u_n$, whenever $u_n \in MF^+(E)\ \forall n \in \mathbb{N}$

$$\int_E \left(\sum_{n=1}^{\infty} u_n\right) = \sum_{n=1}^{\infty} \int_E u_n$$

(ii) $\displaystyle\int_{\bigcup_{n\in\mathbb{N}}^{\circ} E_n} f = \sum_{n=1}^{\infty} \int_{E_n} f$, whenever $f \in \mathcal{MF}^+\left(\bigcup_{n\in\mathbb{N}} E_n\right)$, $E_n \in \mathfrak{M}$ & $E_n \cap E_m = \emptyset\ \forall m \neq n$.

Here $\sum_{n=1}^{\infty} u_n$ is meant to be the function such that

$$\left(\sum_{n=1}^{\infty} u_n\right)(x) = \sum_{n=1}^{\infty} u_n(x) \quad \left(= \lim_{k\to\infty} \sum_{n=1}^{k} u_n(x) = \lim_n s_n(x) \text{ where } s_n = u_1 + \dots + u_n \right)$$

Note that each $s_n \in \mathcal{MF}^+(E)$ and $s_n \uparrow s := \sum_{i=1}^{\infty} u_i$

By MCT, $\displaystyle\int_E s = \lim_n \int s_n = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \int_E u_i \right) = \sum_{i=1}^{\infty} \int_E u_i$

This proves (i). For (ii), note that

$$\int_{\bigcup_{i\in\mathbb{N}}^{\circ} E_i} f = \int \left(f \chi_{\bigcup_{i=1}^{\circ\infty} E_i}\right) = \int f \sum_{i=1}^{\infty} (\chi_{E_i}) = \int \sum_{i=1}^{\infty} (f\chi_{E_i})$$

$$\overset{(i)}{=} \sum_{i=1}^{\infty} \int (f\chi_{E_i}) = \sum_{i=1}^{\infty} \int_{E_i} f, \quad \text{QED.}$$

Th^5 (Absolute Continuity — "ABC" theorem)

Let $f \in \mathcal{MF}^+(\mathbb{R})$ & suppose that $\int_{\mathbb{R}} f < +\infty$.

Then $A \mapsto \int_A f$ is absolutely continuous on $\mathfrak{M}$ (of course, can also be stated for E rather than $\mathbb{R}$):

$\forall \varepsilon > 0\ \exists \delta > 0$ such that

$$m(A) < \delta \Rightarrow \int_A f < \varepsilon .$$

Proof. Easy if f is bounded, so motivates to consider $f \wedge n$ $(n \in \mathbb{N})$. Note that $f \wedge n \in \mathcal{M}^+(\mathbb{R})$ and $f \wedge n \uparrow_n f$ ptwisely so $\lim_n \int (f \wedge n) = \int f$ by the MC theo.

Take $N \in \mathbb{N}$ s.t. $\left| \int (f \wedge N) - \int f \right| < \varepsilon/2$.
Since $0 \leq f - (f \wedge N)$ it follows that

$$\int_A (f - f \wedge N) \leq \int_{\mathbb{R}} (f - f \wedge N) < \varepsilon/2 \quad \forall A \in \mathfrak{M},$$

while

$$\int_A (f \wedge N) \leq \int_A N = N \cdot m(A) \quad \forall A \in \mathfrak{M}$$

Thus, letting $\delta := \frac{\varepsilon/2}{N}$, if $m(A) < \delta$ one has

$$0 \leq \int_A f = \int_A \left[(f - f \wedge N) + (f \wedge N) \right] < \varepsilon/2 + N\delta = \varepsilon .$$